A Simple Model of Competition Between Teams: Online Appendix

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In this Online Appendix we apply the analysis given in Propositions 1 of Esteban and Ray (2001) to our setting and show that it does not imply our results, namely Theorem 1. Proposition 1 of Esteban and Ray (2001) shows that within a fixed equilibrium, a team's effort is increasing in its size, and hence a team's probability of winning is also increasing in its size. The proof is based on the comparative statics derived from totally differentiating the equilibrium first-order condition with respect to team size.

To make our setting as similar as possible to the model of Esteban and Ray (2001), consider modifying our model such that team performance is the sum of individual efforts, but the individual cost function $c(e)$ is increasing, twice continuously differentiable and possibly nonlinear. Also for simplicity, assume the distribution of v is uniform. Thus $P_X(v) = \sum_{i=1}^{n_X} e_i(v)$.

The first-order condition of an in-team symmetric monotone PBNE is

$$G'_Y(P_X(v))v = c'(P_X(v)/n_X) \tag{0.1}$$

Recall that in our original model with nonlinear additively separable team performance and linear costs, the first-order condition is

$$G'_Y(P_X(v))v = \frac{1}{h'(h^{-1}(P_X(v)/n_X))}.$$

Since $G_Y(z) = F(P_Y^{-1}(z))$ where F is the uniform distribution, the first-order condition 0.1 can be rewritten as

$$\Big(P_Y^{-1}\Big)'\Big(P_X(v)\Big)v = c'\Big(P_X(v)/n_X\Big). \tag{0.2}$$

Because of incomplete information, first-order condition 0.2 contains a continuum of equations, instead of just one equation as in Esteban and Ray (2001).

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The approach analogous to Esteban and Ray (2001) would be to totally differentiate 0.2 with respect to n_X for every v. However, naively doing this is mistaken because as n_X changes the total number of participants in the competition changes so the competition changes fundamentally. Consequently, we get *across*-equilibria comparative statics instead of the intended *within*-equilibrium exercise.[1]

To derive within-equilibrium comparative statics in our framework, suppose $N = n_B + n_S$ is fixed. For any $n \in (0, N)$ let $(P_{(n)}, P_{(N-n)})$ be the solution to the boundary value problem given by

$$
\begin{aligned}
\Big(P_{(N-n)}^{-1}\Big)'\Big(P_{(n)}(v)\Big)v &= c'\Big(P_{(n)}(v)/n\Big) \\
\Big(P_{(n)}^{-1}\Big)'\Big(P_{(N-n)}(v)\Big)v &= c'\Big(P_{(N-n)}(v)/(N-n)\Big) \\
P_{(N-n)}(1) &= P_{(n)}(1), \\
\min\{P_{(N-n)}^{-1}(0), P_{(n)}^{-1}(0)\} &= 0.
\end{aligned}
$$

$(P_{(n)}, P_{(N-n)})$ is unique for any n by the proof of Lemma 2. Clearly, $(P_{(n)}, P_{(N-n)}) = (P_B, P_S)$ whenever $n_B = n$ and $n_S = N - n$. Fixing v, wherever $P_{(n)}(v)$ is continuous in n denote

$$p_{(n)}(v) := \lim_{\epsilon \to 0} \frac{P_{(n+\epsilon)}(v) - P_{(n)}(v)}{\epsilon}.$$

The result along the same aline as Proposition 1 of Esteban and Ray (2001) would be that $p_{(n)}(v) \geq 0$ for any $n \in (0, N)$ if c is convex. If this result holds then clearly $P_{(n_B)}(v) > P_{(n_S)}(v)$ for any $v \in (0, 1)$ and hence the bigger team wins with higher probability.

To conduct this comparative statics exercise, denote $Q_{(n)}(x) := \Big(P_{(N-n)}^{-1}\Big)'(x)$ and

$$q_{(n)}(x) := \lim_{\epsilon \to 0} \frac{Q_{(n+\epsilon)}(x) - Q_{(n)}(x)}{\epsilon}.$$

Differentiating this equation with respect to n yields

$$q_{(n)}\Big(P_{(n)}(v)\Big)p_{(n)}(v)v = c''\Big(P_{(n)}(v)/n\Big)\Big(\frac{1}{n}p_{(n)}(v) - \frac{1}{n^2}P_{(n)}(v)\Big).$$

[1] Esteban and Ray (2001) does not have this issue because in their setting each team's first-order condition depends only on *its own size* and the *total effort* from all teams, but does not explicitly depend on sizes of other teams.

Rearranging, we have

$$p_{(n)}(v) = \frac{c''\Big(P_{(n)}(v)/n\Big)P_{(n)}(v)}{nc''\Big(P_{(n)}(v)/n\Big) - n^2 v q_{(n)}\Big(P_{(n)}(v)\Big)}. \tag{0.3}$$

In contrast to Esteban and Ray (2001), the sign of $p_{(n)}(v)$ is not pinned down by equation 0.3. In particular, it depends on the sign and magnitude of $q_{(n)}$, which is not obvious.

References

Esteban, Joan and Debraj Ray (2001), "Collective action and the group size paradox." American Political Science Review, 95, 663–672.